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# Universal scaling behaviour of directed percolation and the pair contact process in an external field

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## Abstract

We consider the scaling behaviour of directed percolation and of the pair contact process with a conjugated field. In particular we determine numerically the equation of state and show that both models are characterized by the same universal scaling function. Furthermore we derive the equation of state for the pair contact process within a mean-field approach which again agrees with the mean-field equation of state of the directed percolation universality class.

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## 1. Introduction

In this paper we consider the scaling behaviour of direction percolation (DP) and of the pair contact process (PCP) numerically in  $1 + 1$  dimensions and analytically within a mean-field approximation (see for a review [1]). Both models display a continuous phase transition from an active to an inactive state. In contrast to the unique absorbing state of DP, the PCP is characterized by infinitely many absorbing states. Despite this difference it was observed in numerical simulations that the critical exponents describing the steady-state scaling behaviour of the PCP agree well with those of DP [2, 3]. This result is in agreement with a renormalization group analysis of a phenomenological field theory approach of the PCP which reduces in the steady state to the corresponding DP equation [4].

In this work we consider the steady-state scaling behaviour of DP and the PCP in an external field which is conjugated to the order parameter. In particular we examine the scaling behaviour of the order parameter (equation of state) as well as of the order parameter fluctuations. In contrast to previous works on the PCP, we do not restrict our attention to the critical exponents but consider the so-called universal scaling functions too. In this way we are able to show that both models are characterized by the same universal scaling behaviour,

i.e. the steady-state scaling behaviour of the PCP belongs to the universality class of directed percolation.

In the next section we briefly review the scaling behaviour of DP and define the scaling functions. We consider then in section 3 for the first time the PCP in a conjugated field and show that the scaling behaviour is characterized by the DP critical exponents. The universal scaling behaviour of DP and the PCP is presented in section 4. The obtained universal equation of state is compared to the results of a two loop renormalization group approach of the corresponding Langevin equation [5]. Finally we derive the equation of state of the PCP within a mean-field approximation. Therefore, we consider the PCP with particle creation at randomly selected sites. This random neighbour interaction suppresses long range correlations and the model is analytically tractable.

## 2. Directed percolation

In order to examine the scaling behaviour of the (1 + 1)-dimensional DP universality class we consider the directed site percolation process using the Domany–Kinzel automaton [6]. It is defined on a diagonal square lattice and evolves by parallel update according to the following rules. A site at time  $t$  is occupied with probability  $p_2$  ( $p_1$ ) if both (only one) backward sites (time  $t - 1$ ) are occupied. If both backward sites are empty a spontaneous particle creation takes place with probability  $p_0$ . Directed site percolation corresponds to the choice  $p_1 = p_2 = p$  and  $p_0 = 0$ . The propagation probability  $p$  is the control parameter of the phase transition, i.e. below a critical value  $p_c$  the activity ceases, and the system is trapped forever in the absorbing state (empty lattice). On the other hand, a non-zero density of (active) particles  $\rho_a$  is found for  $p > p_c$ . The best estimate of the critical value of directed site percolation is  $p_c = 0.705\,489(4)$  [7] and we use this value throughout this work.

The density of active sites  $\rho_a$  is interpreted as the order parameter of the absorbing phase transition and vanishes at the critical point according to

$$\rho_a \propto \delta p^\beta \quad (1)$$

with  $\delta p = (p - p_c)/p_c$ . Furthermore the order parameter fluctuations  $\Delta\rho_a = L(\langle\rho_a^2\rangle - \langle\rho_a\rangle^2)$  diverge as

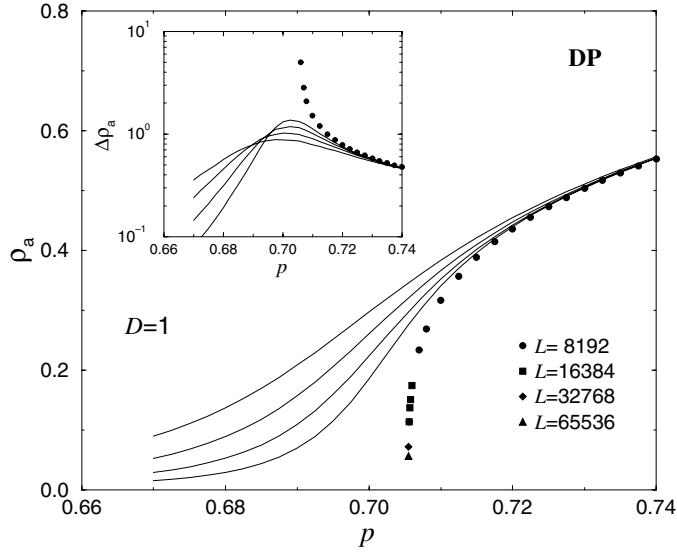
$$\Delta\rho_a \propto \delta p^{-\gamma'}. \quad (2)$$

The fluctuation exponent  $\gamma'$  obeys the scaling relation  $\gamma' = \nu_\perp - 2\beta$ , where  $\nu_\perp$  describes the divergence of the spatial correlation length at the critical point. The critical behaviour of the order parameter is shown in figure 1. The data are obtained from numerical simulations of systems with periodic boundary conditions. Considering various system sizes  $L \leq 131\,072$  we take care that our results are not affected by finite-size effects.

Similar to equilibrium phase transitions it is possible in DP to apply an external field  $h$  that is conjugated to the order parameter. Being a conjugated field it has to destroy the absorbing phase and the corresponding linear response function has to diverge at the critical point, i.e.

$$\chi = \frac{\partial\rho_a}{\partial h} \rightarrow \infty. \quad (3)$$

In DP the external field is implemented (see for instance [1]) as a spontaneous creation of particles ( $p_0 = h > 0$ ). Clearly the spontaneous creation of particles destroys the absorbing state and, therefore, the absorbing phase transition at all. Figure 1 shows how the external field results in a smoothing of the zero-field order parameter curve. The inset displays that the fluctuations peak for finite fields. Approaching the transition point ( $h \rightarrow 0$ ) this peak becomes a divergence signalling the critical point.



**Figure 1.** The directed percolation order parameter  $\rho_a$  as a function of the particle density for zero field (symbols) and for various values of the external field ( $h = 10^{-4}, 2 \times 10^{-4}, 5 \times 10^{-4}, 10^{-3}$ ) (lines). The inset displays the order parameter fluctuations  $\Delta\rho_a$  for zero field (symbols) and for various values of the external field  $h$  (lines).

Close to the transition point the order parameter and its fluctuations obey the following scaling ansätze:

$$\rho_a(\delta p, h) \sim \lambda \tilde{r}_{\text{sDP}}(\delta p \lambda^{-1/\beta}, h \lambda^{-\sigma/\beta}) \quad (4)$$

$$\Delta\rho_a(\delta p, h) \sim \lambda^{\gamma'} \tilde{d}_{\text{sDP}}(\delta p \lambda, h \lambda^{\sigma}) \quad (5)$$

with the scaling functions  $\tilde{r}_{\text{sDP}}$  and  $\tilde{d}_{\text{sDP}}$ , where the index sDP denotes site directed percolation. The scaling function of the order parameter corresponds to the equation of state, and we recover equation (1) by setting  $\delta p \lambda^{-1/\beta} = 1$  at  $h = 0$ . On the other hand, setting  $h \lambda^{-\sigma/\beta} = 1$  leads to

$$\rho_a(\delta p = 0, h) \propto h^{\beta/\sigma} \quad (6)$$

at  $\delta p = 0$ . Analogously we get the scaling behaviour of the fluctuations at  $p_c$

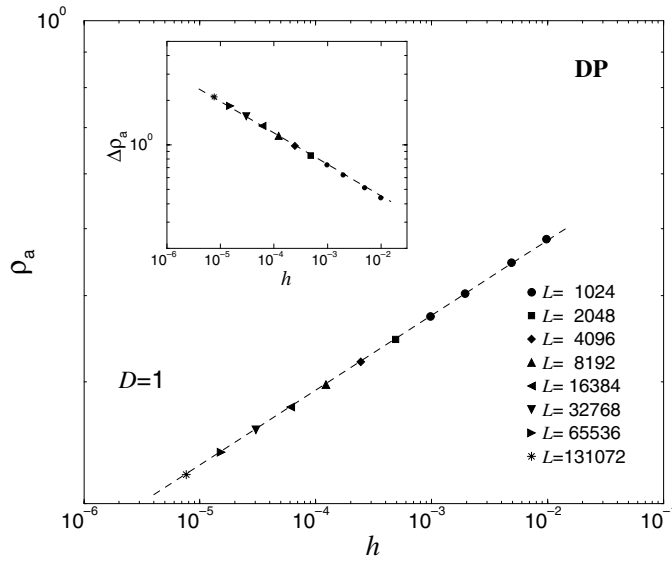
$$\Delta\rho_a(\delta p = 0, h) \propto h^{-\gamma'/\sigma} \quad (7)$$

as well as equation (2). The field dependence of the order parameter and of its fluctuations at  $p_c$  is plotted in figure 2.

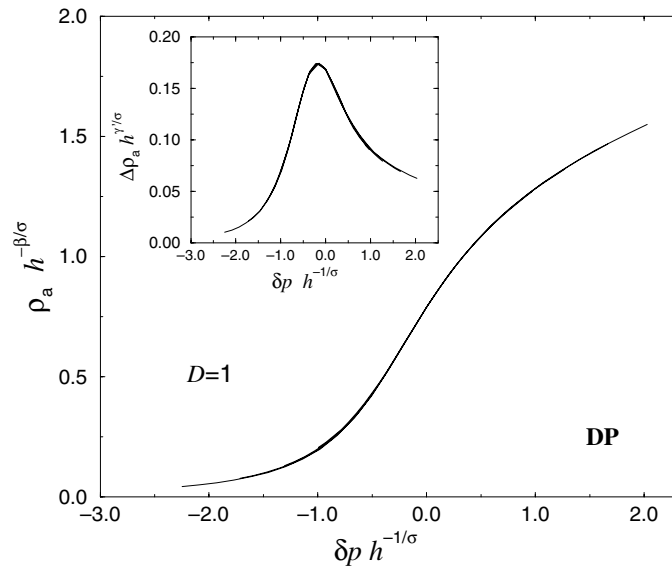
The above scaling forms imply that curves corresponding to different values of the external field collapse to a single one if  $\rho_a h^{-\beta/\sigma}$  and  $\Delta\rho_a h^{\gamma'/\sigma}$  are considered as functions of the rescaled control parameter  $\delta p h^{-1/\sigma}$ . Using the best known estimates of the critical exponents  $\beta = 0.276486$ ,  $\sigma = 2.554216$  and  $\gamma' = 0.543882$  [8], we get beautiful data collapses of our numerical data (see figure 3).

It is worth mentioning that the validity of the scaling behaviour of the equation of state (4) implies the required singularity of the linear response function (3). Using the fact that the susceptibility is defined as the derivative of the order parameter with respect to the conjugated field we get

$$\chi(\delta p, h) = \frac{\partial}{\partial h} \rho_a(\delta p, h) \quad (8)$$



**Figure 2.** The field dependence of the order parameter and its fluctuations (inset) at the critical value  $p_c$  for directed percolation. The dashed lines correspond to the expected power-law behaviour (6), (7).

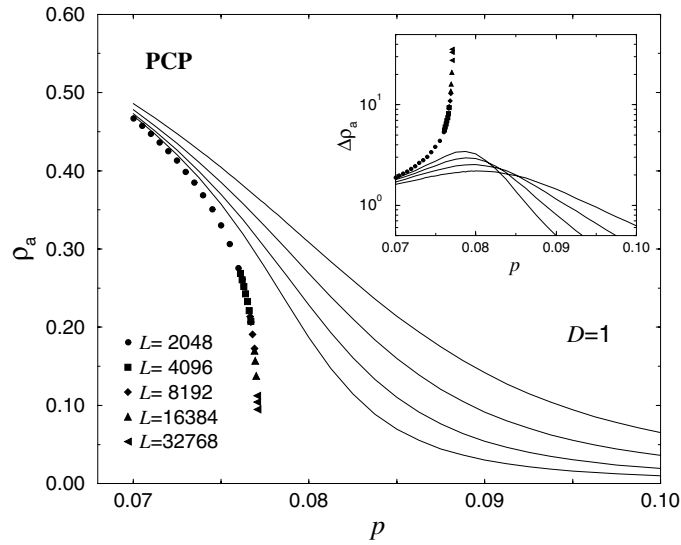


**Figure 3.** The scaling plot of the order parameter and its fluctuations (inset) for directed percolation.

$$\sim \lambda^{1-\sigma/\beta} \tilde{c}_{s,DP}(\delta p \lambda^{-1/\beta}, h \lambda^{-\sigma/\beta}). \tag{9}$$

Thus we get for the critical behaviour

$$\chi(\delta p, h) = \left. \frac{\partial}{\partial h} \rho_a(\delta p, h) \right|_{h \rightarrow 0} \propto \delta p^{-\gamma} \tag{10}$$



**Figure 4.** The pair contact process order parameter  $\rho_a$  as a function of the particle density for zero field (symbols) and for various values of the external field from ( $h = 10^{-4}, 2 \times 10^{-4}, 5 \times 10^{-4}, 10^{-3}$ ) (lines). The inset displays the order parameter fluctuations  $\Delta\rho_a$  for zero field (symbols) and for various values of the external field  $h$  (lines).

and

$$\chi(\delta p, h) = \left. \frac{\partial}{\partial h} \rho_a(\delta p, h) \right|_{\delta p \rightarrow 0} \propto h^{-\gamma/\sigma}. \quad (11)$$

Here the susceptibility exponent  $\gamma$  is given by the scaling relation

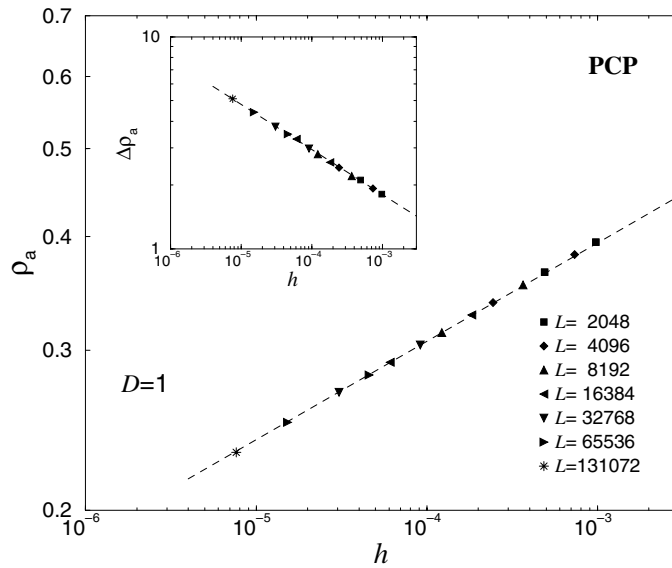
$$\gamma = \sigma - \beta \quad (12)$$

which corresponds to the well-known Widom equation of equilibrium phase transitions. Note that in contrast to the scaling behaviour of equilibrium phase transitions the non-equilibrium absorbing phase transition of DP is characterized by  $\gamma \neq \gamma'$ .

Again, the validity of the scaling ansatz of the order parameter (4) can be used to check that the implemented field satisfies the condition (3), i.e. to check whether the field is conjugated to the order parameter or not. We apply this procedure in the following section where we examine the scaling behaviour of the PCP.

### 3. Pair contact process

The PCP as introduced by Jensen [9] is one of the simplest models with infinitely many absorbing states showing a continuous phase transition. At time  $t$ , sites on a lattice of length  $L$  with periodic boundary conditions can either be occupied ( $n_i(t) = 1$ ) or empty ( $n_i(t) = 0$ ). Pairs of adjacent occupied sites  $i, i+1$ , linked by an active bond, annihilate each other with rate  $p$  or create an offspring with rate  $1-p$  at either site  $i-1$  or  $i+2$  provided the target site is empty. The density of active bonds  $\rho_a$  is the order parameter of a continuous phase transition from an active state for  $p < p_c$  to an inactive absorbing state without particle pairs. The behaviour of the PCP order parameter and its fluctuations are plotted in figure 4. The data are obtained from simulations on various system sizes  $L \leq 131\,072$  with periodic boundary conditions.



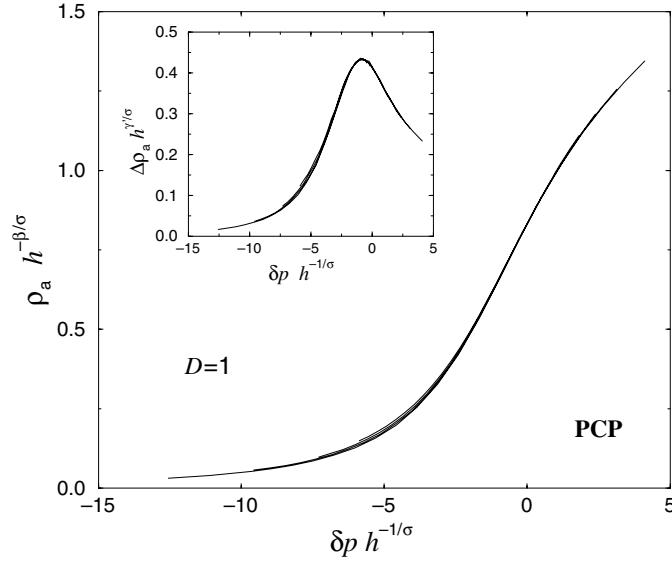
**Figure 5.** The field dependence of the order parameter and its fluctuations (inset) at the critical value  $p_c$  for the pair contact process. The dashed lines correspond to the expected power-law behaviour (6), (7).

Our analysis reveals that the critical value is  $p_c = 0.077\,093(3)$  which agrees with the value  $p_c = 0.077\,090(5)$  [10] obtained from a finite-size scaling analysis of the lifetime distribution.

In contrast to DP there is no unique absorbing state (empty lattice) but infinitely many, as any configuration with only isolated inactive particles is absorbing. Thus in the thermodynamic limit the system will be trapped in one of an infinite number of absorbing configurations for  $p > p_c$ . Despite the different structure of the absorbing states the steady-state scaling behaviour of the PCP is believed to be characterized by the DP critical exponents  $\beta$ ,  $\gamma'$ ,  $\gamma$ , etc. On the other hand, the dynamical scaling behaviour, associated with activity spreading of a localized seed, depends on the details of the system preparation [3].

Recently, Dickman *et al* [11] considered the PCP with an external field that randomly creates isolated particles. Thus the external field couples to the particle density but not to the order parameter itself, i.e. the external field is not conjugated to the order parameter. The authors observe that the external field shifts the critical values  $p_c$  continuously and that the critical exponents are unaffected by the presence of the particle source. In order to investigate the PCP in a conjugated field the implementation of the external field of [11] has to be modified. Several modifications of the external field are possible. For instance, in absorbing phase transitions with particle conservation [12], the conjugated field triggers movements of inactive particles which can be activated in this way [13].

As shown below, spontaneous particle creation with rate  $h$  acts as a conjugated field analogous to DP. Figure 4 shows how the spontaneous particle creation smooths the critical zero field curves similar to the DP behaviour (see figure 1). We simulated the PCP at the critical value  $p_c$  for various fields. The order parameter and its fluctuations as a function of the external field  $h$  are presented in figure 5. Approaching the transition point,  $\rho_a$  and  $\Delta\rho_a$  scale according to equations (6) and (7) where the exponents  $\beta/\sigma$  and  $\gamma'/\sigma$  agree with the DP values.



**Figure 6.** The scaling plot of the order parameter and its fluctuations (inset) for the pair contact process.

Furthermore, we assume that the order parameter and order parameter fluctuations analogous to DP obey the scaling forms

$$\rho_a(\delta p, h) \sim \lambda \tilde{r}_{\text{PCP}}(\delta p \lambda^{-1/\beta}, h \lambda^{-\sigma/\beta}) \quad (13)$$

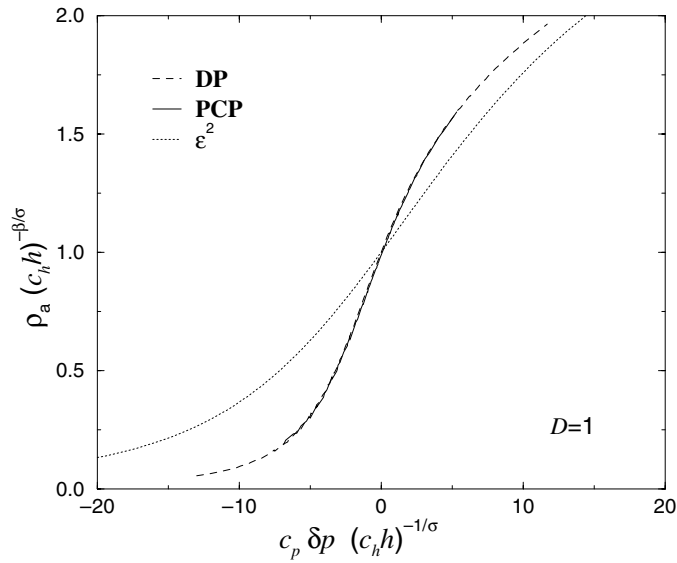
$$\Delta\rho_a(\delta p, h) \sim \lambda^{\gamma'} \tilde{d}_{\text{PCP}}(\delta p \lambda, h \lambda^\sigma) \quad (14)$$

where the distance to the critical point is now given by  $\delta p = (p_c - p)/p_c$ . Using the DP values of the critical exponents  $\beta$ ,  $\sigma$  and  $\gamma'$  we get convincing data collapses (see figure 6). As pointed out above, the validity of the scaling ansatz (13) implies the singular behaviour of the linear response function (3), i.e. the spontaneous particle creation in the PCP can be interpreted as an external field conjugated to the order parameter. Furthermore the data collapse confirms again that the steady-state scaling behaviour of the PCP is characterized by the DP exponents.

#### 4. Universal scaling behaviour

One of the most striking features of critical phenomena is the concept of universality, i.e. close to the critical point the scaling behaviour depends only on a few fundamental parameters whereas the interaction details of the systems do not alter the scaling behaviour. In the case of systems with short range interactions these parameters are the symmetry of the order parameter and the dimensionality of space  $D$  [14, 15]. Classical examples of such universal behaviour are for instance the coexistence curve of liquid–vapour systems [16] and the equation of state in ferromagnetic systems [17]. In the case of absorbing phase transitions, the corresponding universality hypothesis states that systems exhibiting a continuous phase transition to a unique absorbing state generally belong to the universality class of directed percolation [18, 19]. A different scaling behaviour is observed only in systems with additional symmetries (such as parity conservation or particle conservation) or in systems with quenched randomness.





**Figure 7.** The universal order parameter scaling function  $\tilde{R}_{\text{DP}}(x, 1)$  of the universality class of directed percolation. The dotted line corresponds to the result of a two-loop renormalization group analysis of the Langevin equation [5] (see the text).

Following the concept of universality, two models belong to the same universality class if the critical exponents and the universal scaling functions are identical. The universal scaling functions  $\tilde{R}_{\text{DP}}$  and  $\tilde{D}_{\text{DP}}$  of the DP universality class can be easily determined by introducing non-universal metric factors  $c_i$  and  $d_i$  (see, e.g., [20]) for each scaling argument in the scaling functions (4), (5), (13) and (14), i.e.

$$\rho_a(\delta p, h) \sim \lambda \tilde{R}_{\text{DP}}(c_1 \delta p \lambda^{-1/\beta}, c_2 h \lambda^{-\sigma/\beta}) \quad (15)$$

$$\Delta \rho_a(\delta p, h) \sim \lambda^{\gamma'} \tilde{D}_{\text{DP}}(d_1 \delta p \lambda, d_2 h \lambda^\sigma). \quad (16)$$

Using the normalizations  $\tilde{R}_{\text{DP}}(1, 0) = \tilde{R}_{\text{DP}}(0, 1) = 1$  and  $\tilde{D}_{\text{DP}}(1, 0) = \tilde{D}_{\text{DP}}(0, 1) = 1$ , the non-universal metric factors can be determined from the amplitudes of

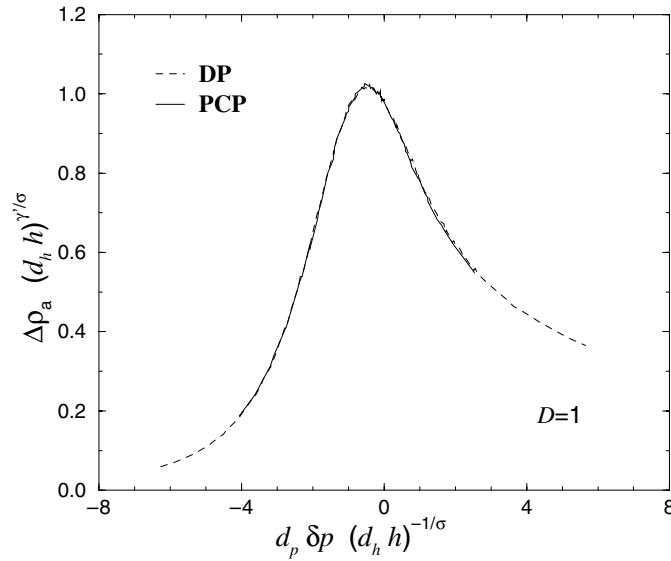
$$\rho_a(\delta p, h = 0) \sim (c_1 \delta p)^\beta \quad \rho_a(\delta p = 0, h) \sim (c_2 h)^{\beta/\sigma} \quad (17)$$

and

$$\Delta \rho_a(\delta p, h = 0) \sim (d_1 \delta p)^{-\gamma'} \quad \Delta \rho_a(\delta p = 0, h) \sim (d_2 h)^{-\gamma'/\sigma}. \quad (18)$$

Like the value of the critical point  $p_c$ , the non-universal metric factors may depend on the details of the considered system, e.g., the lattice structure, the boundary conditions, the used update scheme, etc. In the case of directed site percolation in the Domany–Kinzel automaton we have obtained the values  $c_1^{\text{sDP}} = 2.45$ ,  $c_2^{\text{sDP}} = 0.112$ ,  $d_1^{\text{sDP}} = 71.6$  and  $d_2^{\text{sDP}} = 3984$ . For the PCP on a square lattice we have determined the values  $c_1^{\text{PCP}} = 0.665$ ,  $c_2^{\text{PCP}} = 0.181$ ,  $d_1^{\text{PCP}} = 3.21$  and  $d_2^{\text{PCP}} = 62.11$ .

Analogous to the previous scaling analysis we set  $c_2 h \lambda^{-\sigma/\beta} = 1$  and consider for both models the rescaled order parameter  $\rho_a(c_2 h)^{-\beta/\sigma}$  as a function of the rescaled control parameter  $c_1 \delta p (c_2 h)^{-1/\sigma}$  as well as the rescaled order parameter fluctuations  $\Delta \rho_a(d_2 h)^{\gamma'/\sigma}$  as a function of  $d_1 \delta p (d_2 h)^{-1/\sigma}$ , respectively. The corresponding data are presented in figures 7 and 8. In both cases we get a perfect data collapse of the curves showing that the one-dimensional PCP steady-state scaling behaviour belongs to the universality class of directed percolation.



**Figure 8.** The universal scaling function  $\tilde{D}_{\text{DP}}(x, 1)$  of the order parameter fluctuations of the universality class of directed percolation.

In addition to the universal scaling function  $\tilde{R}_{\text{DP}}(x, 1)$ , we plot in figure 7 the corresponding curve of an  $\epsilon$ -expansion obtained from a renormalization group analysis of a Langevin equation (see equation (25), next section). Using the parametric representation [21, 22] of the absorbing phase transition, Janssen *et al* showed recently that the equation of state is given by the remarkably simple scaling function [5]

$$H(\theta) = \theta(2 - \theta) + O(\epsilon^3) \quad (19)$$

where  $\epsilon$  denotes the distance to the upper critical dimension  $D_c = 4$ , i.e.  $\epsilon = D_c - D$ . Here the scaling behaviour of the quantities  $\rho_a$ ,  $\delta p$  and  $h$  is transformed to the variables  $R$  and  $\theta$  through the relations

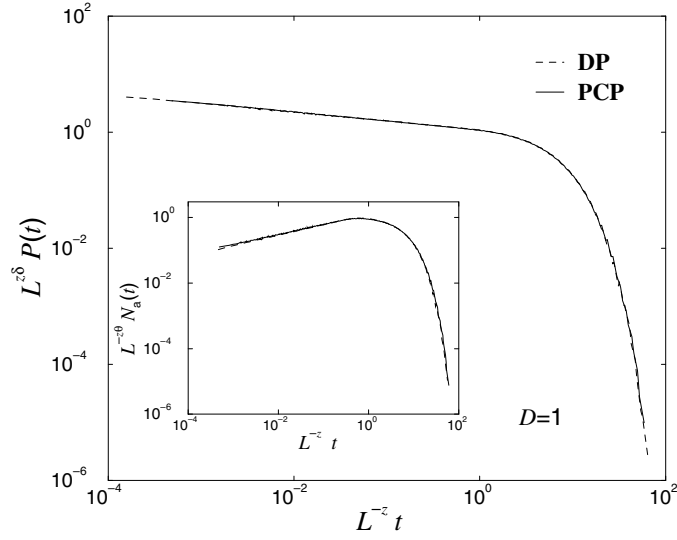
$$b\delta p = R(1 - \theta) \quad \rho_a = R^\beta \frac{\theta}{2}. \quad (20)$$

The equation of state is given by

$$ah = \left(\frac{R^\beta}{2}\right)^\delta H(\theta) \quad (21)$$

with the metric factors  $a$  and  $b$ . The whole phase diagram is described by the parameter range  $R \geq 0$  and  $\theta \in [0, 2]$ . Using the equations (19–21) we calculated the corresponding universal function and compare it in figure 7 with the numerically obtained universal scaling function  $\tilde{R}_{\text{DP}}(x, 1)$ . As can be seen the significant difference indicates that the  $O(\epsilon^3)$  corrections to (19) are relevant, i.e. higher orders than  $O(\epsilon^2)$  are necessary to describe the scaling behaviour of directed percolation. Of course we expect that this difference will decrease with increasing dimension, i.e. for  $\epsilon \rightarrow 0$ .

Before closing this section we briefly mention that the dynamical scaling behaviour of the PCP belongs to the DP universality class too if the spreading of a localized seed is considered at the so-called natural particle density [3]. Examining spreading activity one usually considers the survival probability  $P_a$  of the activity as well as how the number of



**Figure 9.** The DP universal finite-size scaling functions  $\tilde{P}$  and  $\tilde{N}$ . Both quantities describe the activity spreading of a localized seed (see the text).

active particles  $N_a = L\rho_a$  increases in time. In the case of DP the simulations are started with a single seed on an empty lattice. For the PCP an absorbing state at  $p_c$  is prepared to which a particle is added in order to create one seed (one active pair). At the critical point the following power-law behaviours are expected:

$$N_a \propto t^\theta \quad P_a \propto t^{-\delta}. \quad (22)$$

Finite-system sizes limit these power-law behaviours and  $P_a$  and  $N_a$  obey the finite-size scaling ansätze

$$N_a(\delta p = 0, L) \sim \lambda \tilde{n}(\lambda^{-1/\theta} t, \lambda^{-1/\theta z} L) \quad (23)$$

$$P_a(\delta \rho = 0, L) \sim \lambda \tilde{p}(\lambda^{1/\delta} t, \lambda^{1/\delta z} L). \quad (24)$$

where  $z$  denotes the dynamical exponent. Analogous to the above analysis the universal scaling curves  $\tilde{N}$  and  $\tilde{P}$  are obtained by introducing appropriate non-universal metric factors. Using the values  $z = 1.580745$ ,  $\theta = 0.313686$  and  $\delta = 0.159464$  [8] we get convincing data collapses and the universal scaling functions are plotted in figure 9.

## 5. Mean-field scaling behaviour

The mean-field equation of state of DP can be easily derived from the corresponding Langevin equation

$$\partial_t \rho_a = \delta p \rho_a - \lambda \rho_a^2 + \kappa h + D \nabla^2 \rho_a + \eta \quad (25)$$

which describes the order parameter field  $\rho_a(\underline{x}, t)$  on a mesoscopic scale (see [1] for a detailed discussion). Here  $D$  denotes the diffusion constant,  $\eta$  denotes a multiplicative noise term with the correlator

$$\langle \eta(\underline{x}, t) \eta(\underline{x}', t') \rangle = \Gamma \rho_a(\underline{x}, t) \delta^d(\underline{x} - \underline{x}') \delta(t - t') \quad (26)$$

**Table 1.** The configuration of a PCP lattice before ( $\mathcal{C}$ ) and after ( $\mathcal{C}'$ ) an event. Only the sites left and right of those changed by particle creation (top), pair annihilation (middle) or particle creation due to the external field (bottom) are shown. Empty sites are marked by  $\circ$ , and occupied sites by  $\bullet$ . Here,  $\Delta n_a$  denotes the change of the number of active bonds,  $\Delta n_i$  the respective change of inactive bonds,  $\Delta n_e$  that of empty bonds and  $P$  is the corresponding probability of the event if spatial correlations are neglected.

$\mathcal{C}$	$\mathcal{C}'$	$\Delta n_a$	$\Delta n_i$	$\Delta n_e$	$p(\mathcal{C} \rightarrow \mathcal{C}')$
$\bullet \circ \bullet$	$\bullet \bullet \bullet$	+2	-2	0	$(1-p)\rho_a\rho_i^2$
$\bullet \circ \circ$	$\bullet \bullet \circ$	+1	0	-1	$(1-p)\rho_a\rho_i\rho_e$
$\circ \circ \bullet$	$\circ \bullet \bullet$	+1	0	-1	$(1-p)\rho_a\rho_i\rho_e$
$\circ \circ \circ$	$\circ \bullet \circ$	0	+2	-2	$(1-p)\rho_a\rho_e^2$
$\bullet \bullet \bullet \bullet$	$\bullet \circ \circ \bullet$	-3	+2	+1	$p\rho_a^3$
$\circ \bullet \bullet \bullet$	$\circ \circ \circ \bullet$	-2	0	+2	$p\rho_a^2\rho_i$
$\bullet \bullet \bullet \circ$	$\bullet \circ \circ \circ$	-2	0	+2	$p\rho_a^2\rho_i$
$\circ \bullet \bullet \circ$	$\circ \circ \circ \circ$	-1	-2	+3	$p\rho_a\rho_i^2$
$\bullet \circ \bullet$	$\bullet \bullet \bullet$	+2	-2	0	$h\rho_i^2$
$\bullet \circ \circ$	$\bullet \bullet \circ$	+1	0	-1	$h\rho_i\rho_e$
$\circ \circ \bullet$	$\circ \bullet \bullet$	+1	0	-1	$h\rho_i\rho_e$
$\circ \circ \circ$	$\circ \bullet \circ$	0	+2	-2	$h\rho_e^2$

and  $\lambda > 0$ ,  $\kappa > 0$  and  $\Gamma > 0$  are certain coupling constants. Neglecting spatial correlations and fluctuations ( $D = 0$  and  $\eta = 0$ ) one gets for the steady-state behaviour ( $\partial_t \rho_a = 0$ )

$$\delta p \rho_a - \lambda \rho_a^2 + \kappa h = 0 \quad (27)$$

from which it is easy to derive the universal scaling function

$$\rho_a = \tilde{R}_{\text{DP}}(c_1^{\text{DP}} \delta p, c_2^{\text{DP}} h) = \frac{c_1^{\text{DP}} \delta p}{2} + \sqrt{c_2^{\text{DP}} h + \left(\frac{c_1^{\text{DP}} \delta p}{2}\right)^2} \quad (28)$$

where the non-universal metric factors are given by  $c_1^{\text{DP}} = 1/\lambda$  and  $c_2^{\text{DP}} = \kappa/\lambda$ , respectively. For zero field we get the solutions  $\rho_a = 0$  (absorbing state) and  $\rho_a = c_1^{\text{DP}} \delta p$ , i.e. the mean-field value of the order parameter exponent is  $\beta = 1$ . On the other hand,  $\delta p = 0$  leads to  $\rho_a = (c_2^{\text{DP}} h)^{1/2}$  implying  $\sigma = 2$ . As recently shown the universal scaling function (28) describes not only the scaling behaviour of the DP universality class but it also occurs in the different universality class of absorbing phase transitions with a conserved field [23].

Let us now consider the following modification of the PCP. An active bond produces an offspring with rate  $(1-p)$  at an empty site selected at random. The rules for annihilation and action of the external field remain unchanged. This random neighbour interaction suppresses long range correlations, and the model is, therefore, expected to be characterized by the mean-field scaling behaviour. We denote the density of inactive bonds between an occupied and an empty site as  $\rho_i$ . Bonds between empty sites are denoted as  $\rho_e$ . Normalization requires  $\rho_e + \rho_i + \rho_a = 1$ .

Depending on the sites adjacent to the target site the number of active bonds  $n_a$ , inactive bonds  $n_i$  or empty bonds  $n_e$  is changed. For instance, if the adjacent sites are empty, for which the probability in the absence of correlations is  $\rho_e^2$ , the number of empty bonds decreases by two ( $\Delta n_e = -2$ ). On the other hand, there are two new inactive bonds ( $\Delta n_i = +2$ ). The total probability for this event is  $(1-p)\rho_a\rho_e^2$ . A list of all possible processes and their mean-field probabilities is given in table 1. Thus we obtain rate equations for the expectation values

$E[\Delta n_x]$  of the changes in active, inactive and empty bond numbers. These expectation values are zero in the steady state, i.e.

$$E[\Delta n_x] = \sum_{\Delta n_x} \Delta n_x p(\Delta n_x) = 0 \quad (29)$$

with  $x \in \{a, i, e\}$ . In the case of  $E[\Delta n_a]$  we get

$$E[\Delta n_a] = -3p\rho_a^3 - 4p\rho_a^2\rho_i - p\rho_a\rho_i^2 + 2(1-p)\rho_a\rho_i^2 + 2(1-p)\rho_a\rho_i\rho_e + 2h\rho_i^2 + 2h\rho_i\rho_e = 0 \quad (30)$$

whereas we get for the inactive bonds

$$E[\Delta n_i] = -2(1-p)\rho_a\rho_i^2 + 2(1-p)\rho_a\rho_e^2 + 2p\rho_a^3 - 2p\rho_a\rho_i^2 - 2h\rho_i^2 + 2h\rho_e^2 = 0. \quad (31)$$

Using these equations together with the normalization allows the calculation of the order parameter for zero field ( $h = 0$ ) which yields the non-trivial ( $\rho_a > 0$ ) solution

$$\rho_a = \frac{8 - 3p^2 - 5p - 2\sqrt{2p(1-p)(3p-4)^2}}{9p^2 - 9p + 8}. \quad (32)$$

This solution is valid below the mean-field critical point  $p_c = 8/9$ , whereas the trivial solution  $\rho_a = 0$  is valid for all  $p$  but unstable above  $p_c$ . Expanding (32) around the critical point leads to

$$\rho_a = \frac{3}{8}\delta p + O(\delta p^2) \quad (33)$$

with  $\delta p = (p_c - p)/p_c$ . Thus the mean-field exponent of the PCP is  $\beta = 1$  and the non-universal metric factor  $c_1^{\text{PCP}} = 3/8$ .

In order to obtain the order parameter in the presence of an external field  $h$  equations (30) and (31) are solved for  $\rho_i$  which yields

$$4h + 4\rho_a - 4h\rho_a - 4p\rho_a - 4\rho_a^2 + \left\{ -12p^2\rho_a^2 + (2h + 2\rho_a - 2h\rho_a - 2p\rho_a - 2\rho_a^2 - 2p\rho_a^2)^2 \right\}^{1/2} - \left\{ (-2h - 2\rho_a + 2p\rho_a + 2\rho_a^2 - 2p\rho_a^2)^2 + 4p\rho_a(h + \rho_a - 2h\rho_a - p\rho_a - 2\rho_a^2 + h\rho_a^2 + 2p\rho_a^2 + \rho_a^3) \right\}^{1/2} = 0. \quad (34)$$

To obtain the field dependence of the order parameter, a series expansion around  $h = 0$  at  $p_c = 8/9$  is performed which results in leading order

$$\rho_a = \sqrt{\frac{3}{8}h} \quad (35)$$

i.e. the mean-field values of the PCP are given by  $c_2^{\text{PCP}} = \sqrt{3/8}$  and  $\sigma = 2$ .

Finally we derive the mean-field universal scaling function  $\tilde{R}$  of the PCP. Therefore, we write (34) as a function of the reduced control parameter  $\delta p$  and perform the limits  $\rho_a \rightarrow 0$ ,  $\delta p \rightarrow 0$  and  $h \rightarrow 0$  with the constraint that  $\rho_a/\sqrt{h}$  and  $\rho_a/\delta p$  are finite. Thus we remain in leading order with

$$\frac{3}{8}\delta p\rho_a - \rho_a^2 + \frac{3}{8}h = 0. \quad (36)$$

Solving this equation yields

$$\rho_a = \frac{1}{2}\frac{3}{8}\delta p + \sqrt{\frac{3}{8}h + \left(\frac{1}{2}\frac{3}{8}\delta p\right)^2} = \tilde{R}_{\text{DP}}(c_1^{\text{PCP}}\delta p, c_2^{\text{PCP}}h). \quad (37)$$

Thus we have shown that both models are characterized by the same critical exponents and scaling function, i.e. similar to the (1 + 1)-dimensional case, the mean-field solution of the PCP belongs to the mean-field universality class of directed percolation.

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